Solving DSGE models: an example. Hansens Real Business Cycle Model

IAMA, Lecture 5

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Winter 2006/07

Outline

Outline

The solution strategy

The solution strategy for a model works as follows:

1. **Find the first order necessary conditions**
2. **Calculate the steady state**
3. **Loglinearize around the steady state**
4. **Solve for the recursive law of motion**
5. **Calculate impulse responses and (HP-filtered) moments**

We will execute this strategy, using Hansens real business cycle model as particular example.

Outline

Hansens benchmark Real Business Cycle Model

$$\max E\left[\sum_{t=0}^{\infty}\beta^t(\log c_t - An_t)\right]$$

s.t.

$$c_t + k_t = \bar{\gamma}e^{z_t}k_{t-1}^{\theta}n_t^{1-\theta} + (1-\delta)k_{t-1}$$

and

$$z_t = \rho z_{t-1} + \epsilon_t,\ \ \epsilon_t \sim N(0,\sigma^2)\ i.i.d.$$

where c_t is **consumption**, n_t is **labor**, k_t is **capital**, $\gamma_t = \bar{\gamma}e^{z_t}$ is **total factor productivity (TFP)**.

Hansens benchmark Real Business Cycle Model

Define, for convenience;

$$\textbf{output:} \quad y_t = \bar{\gamma} e^{z_t} k_{t-1}^{\theta} n_t^{1-\theta}$$

$$\textbf{return:} \quad R_t = \theta \frac{y_t}{k_{t-1}} + 1 - \delta$$

See:

1. Hansen, G., "Indivisible Labor and the Business Cycle," Journal of Monetary Economics, 1985, 16, 309-27.
2. Cooley, editor, *Frontiers of Business Cycle Research,* Princeton University Press, 1995.

Outline

Rational expectations

- We assume that the social planner chooses c_t, k_t, n_t etc., using all available information at date t, and forming **rational expectations** about the future.
-

Rational expectations are the mathematical expectations, using all available information

- Rational expectations only ”live in” a model, in which the stochastic nature of all variables is clearly spelled out.

Rational expectations

Example: dice role.

- Dice 1, date t: X_t. Dice 2, date $t+1$: Y_{t+1}. Sum: $S_{t+1} = X_t + Y_{t+1}$.
- $E_{t-1}[S_{t+1}] = 7$. $E_t[S_{t+1}] = 3.5 + X_t$. $E_{t+1}[S_{t+1}] = X_t + Y_{t+1}$.
- E.g. $X_t = 2, Y_{t+1} = 1$. Then $E_{t-1}[S_{t+1}] = 7$, $E_t[S_{t+1}] = 5.5$, $E_{t+1}[S_{t+1}] = 3$.

Example: AR(1)

- $z_{t+1} = \rho z_t + \epsilon_{t+1}, E_t[\epsilon_{t+1}] = 0$.
- Then: $E_t[z_{t+1}] = \rho z_t$.

Outline

Labor lotteries and labor supply

- We assume a very **elastic** labor supply for aggregate labor n_t,
$$u_t = \log(c_t) - An_t$$
- ... which turns out to be needed in order to quantitatively explain observed employment fluctuations.
- However, we typically imagine individual labor elasticity to be small.
- This can be true simultaneously by considering **labor lotteries**.
- Source: Richard Rogerson, ”Indivisible Labor, Lotteries and Equilibrium,” Journal of Monetary Economics; 21(1), January 1988, 3-16.

Labor lotteries and labor supply

- Individual labor supply $\tilde{n}_t$ may be based on some utility function $u(c_t) + v(\tilde{n}_t)$.
- Suppose that
  - labor is **indivisible**: agents either have a job or do not, $\tilde{n}_t = 0$ or $\tilde{n}_t = n^*$.
  - Agents are assigned to jobs according to a lottery, with probability π_t.
  - Shirking, moral hazard etc. are not possible. Unemployment insurance is perfect, and consumption c_t is independent of job status.
  - Total labor supplied: $n_t = \pi_t n^*$
  - Normalization: $v(0) = 0$, $v(n^*)/n^* =: -A < 0$.
- Expected utility:

$$E[u(c_t) + v(\tilde{n}_t)] = u(c_t) + \pi_t v(n^*) = u(c_t) - An_t$$

Outline

Step 1: Find the first-order necessary conditions (FONCS)

- Form the Lagrangian

$$
\begin{aligned}
L = \max E \Bigg[\sum_{t=0}^{\infty} \beta^t \Big((\log c_t - A n_t) \\
- \lambda_t \left(c_t + k_t - \bar{\gamma} e^{z_t} k_{t-1}^{\theta} n_t^{1-\theta} - (1-\delta) k_{t-1} \right) \Big) \Bigg]
\end{aligned}
$$

Find the first-order necessary conditions...

- Differentiate:

$$\begin{aligned}
\frac{\partial L}{\partial c_t} &: \qquad \frac{1}{c_t} = \lambda_t \\
\frac{\partial L}{\partial n_t} &: \qquad A = \lambda_t (1-\theta) \frac{y_t}{n_t} \\
\frac{\partial L}{\partial \lambda_t} &: \qquad c_t + k_t = \bar{\gamma} e^{z_t} k_{t-1}^{\theta} n_t^{1-\theta} + (1-\delta) k_{t-1} \\
\frac{\partial L}{\partial k_t} &: \qquad \lambda_t = \beta E_t [\lambda_{t+1} R_{t+1}]
\end{aligned}$$

The last equation needs explanation.

Differentiating with respect to k_t

- Write out the objective at date t: for the future, one can only form conditional expectations $E_t[\cdot]$. "Telescope" out the Lagrangian:

$$
\begin{aligned}
L = \ldots + \beta^t ((\log c_t - A n_t) \\
- \lambda_t \left(c_t + k_t - \bar{\gamma} e^{z_t} k_{t-1}^{\theta} n_t^{1-\theta} - (1-\delta) k_{t-1} \right)\right) \\
+ E_t \left[\beta^{t+1} ((\log c_{t+1} - A n_{t+1}) \right. \\
\left. - \lambda_{t+1} \left(c_{t+1} + k_{t+1} - \bar{\gamma} e^{z_{t+1}} k_t^{\theta} n_{t+1}^{1-\theta} - (1-\delta) k_t \right)\right) \right] + \ldots
\end{aligned}
$$

- Differentiate with respect to k_t:

$$
0 = \beta^t \lambda_t - E_t \left[\beta^{t+1} \lambda_{t+1} \left(\theta \frac{y_{t+1}}{k_t} + 1 - \delta \right) \right]
$$

Differentiating with respect to k_t

- Sort terms and use

$$R_{t+1} = \theta \frac{y_{t+1}}{k_t} + 1 - \delta$$

to find

$$\boxed{\lambda_t = \beta E_t[\lambda_{t+1} R_{t+1}]}$$

- This equation is called an **Euler equation** and also the **Lucas asset pricing equation**.

Collecting equations

1. First order conditions and a definition:

$$\begin{aligned}
\frac{1}{c_t} &= \lambda_t \\
A &= \lambda_t (1-\theta) \frac{y_t}{n_t} \\
R_t &= \theta \frac{y_t}{k_{t-1}} + 1 - \delta \\
\lambda_t &= \beta E_t[\lambda_{t+1} R_{t+1}]
\end{aligned}$$

2. Technology and Feasibility constraints:

$$\begin{aligned}
y_t &= \bar{\gamma} e^{z_t} k_{t-1}^{\theta} n_t^{1-\theta} \\
c_t + k_t &= y_t + (1-\delta) k_{t-1} \\
z_t &= \rho z_{t-1} + \epsilon_t, \ \epsilon_t \sim N(0, \sigma^2) \ i.i.d.
\end{aligned}$$

Outline

Step 2: Calculate the steady state

At the steady state, all variables are constant.

Take all equations ...

1. First order conditions and a definition:

$$
\begin{aligned}
\frac{1}{c_t} &= \lambda_t \\
A &= \lambda_t (1-\theta) \frac{y_t}{n_t} \\
R_t &= \theta \frac{y_t}{k_{t-1}} + 1 - \delta \\
\lambda_t &= \beta E_t [\lambda_{t+1} R_{t+1}]
\end{aligned}
$$

2. Technology and Feasibility constraints:

$$
\begin{aligned}
y_t &= \bar{\gamma} e^{z_t} k_{t-1}^{\theta} n_t^{1-\theta} \\
c_t + k_t &= y_t + (1-\delta) k_{t-1} \\
z_t &= \rho z_{t-1} + \epsilon_t, \;\; \epsilon_t \sim N(0, \sigma^2) \; i.i.d.
\end{aligned}
$$

... and drop the time subscripts.

1. First order conditions and a definition:

$$\begin{aligned} \frac{1}{\bar{c}} &= \bar{\lambda} \\ A &= \bar{\lambda}(1-\theta)\frac{\bar{y}}{\bar{n}} \\ \bar{R} &= \theta\frac{\bar{y}}{\bar{k}} + 1 - \delta \\ \bar{\lambda} &= \beta\bar{\lambda}\bar{R} \end{aligned}$$

2. Technology and Feasibility constraints:

$$\begin{aligned} \bar{y} &= \bar{\gamma}e^{\bar{z}}\bar{k}^{\theta}\bar{n}^{1-\theta} \\ \bar{c} + \bar{k} &= \bar{y} + (1-\delta)\bar{k} \\ \bar{z} &= \rho\bar{z} \end{aligned}$$

Parameters

1. Calibration: $\theta = 0.4$, $\delta = 0.012$, $\rho = 0.95$, $\sigma_\epsilon = 0.007$, $\beta = 0.987$, $\bar{\gamma} = 1$, A so that $\bar{n} = 1/3$ (see Cooley, *Frontiers...*).
2. Estimation:
    1. GMM: mimics calibration, see Christiano and Eichenbaum, "Current Real-Business Cycle Theories and Aggregate Labor Market Fluctuations," American Economic Review, vol 82, no. 3, 430 - 450.
    2. Maximum Likelihood: see e.g. Leeper and Sims, "Toward a Modern Macroeconomic Model Usable for Policy Analysis," NBER Macroeconomics Annual, 1994, 81 - 177.

With numbers for the parameters, the steady state can be calculated explicitely.

Explicit calculation

From the production function,

$$\bar{y} = \bar{\gamma} e^{\bar{z}} \bar{k}^{\theta} \bar{n}^{1-\theta}$$

we get

$$\bar{y} = \left(\bar{\gamma} e^{\bar{z}} \left(\frac{\bar{y}}{\bar{k}} \right)^{-\theta} \right)^{\frac{1}{1-\theta}} \bar{n}$$

Explicit calculation: $\bar{n}$ given, solve for A.

1. $\bar{R} = \frac{1}{\beta}$
2. $\frac{\bar{y}}{\bar{k}} = \frac{\bar{R} - 1 + \delta}{\theta}$
3. $\bar{y} = \left(\bar{\gamma} e^{\bar{z}} \left(\frac{\bar{y}}{\bar{k}} \right)^{-\theta} \right)^{\frac{1}{1-\theta}} \bar{n}$
4. $\bar{k} = \left(\frac{\bar{y}}{\bar{k}} \right)^{-1} \bar{y}$
5. $\bar{c} = \bar{y} - \delta \bar{k}$
6. $\bar{\lambda} = \frac{1}{\bar{c}}$
7. $A = \bar{\lambda}(1 - \theta)\frac{\bar{y}}{\bar{n}}$

Explicit calculation alternative: A given, solve for $\bar{n}$.

1. $\bar{R} = \frac{1}{\beta}$
2. $\frac{\bar{y}}{\bar{k}} = \frac{\bar{R}-1+\delta}{\theta}$
3. $\frac{\bar{y}}{\bar{n}} = \left(\bar{\gamma} e^{\bar{z}} \left(\frac{\bar{y}}{\bar{k}}\right)^{-\theta}\right)^{\frac{1}{1-\theta}}$
4. $\bar{\lambda} = \frac{A}{(1-\theta)\left(\frac{\bar{y}}{\bar{n}}\right)}$
5. $\frac{\bar{c}}{\bar{k}} = \frac{\bar{y}}{\bar{k}} - \delta$
6. $\bar{c} = \frac{1}{\bar{\lambda}}$
7. $\bar{k} = \frac{\bar{c}}{\left(\frac{\bar{c}}{\bar{k}}\right)}$
8. $\bar{y} = \left(\frac{\bar{y}}{\bar{k}}\right) \bar{k}$

Outline

Step 3: Loglinearize around the steady state

- Replace the dynamic **nonlinear** equations by dynamic **linear** equations.
- Interpretation and calculation are made easier, if the equations are linear in **percent deviations** from the steady state.

The Principle of Loglinearization

For $x \approx 0$,

$$e^x \approx 1 + x$$

For x_t, let $\hat{x}_t = log(x_t/\bar{x})$ be the *log-deviation* of x_t from its steady state. Thus, $100 * \hat{x}_t$ is (approximately) the percent deviation of x_t from $\bar{x}$. Then,

$$x_t = \bar{x}e^{\hat{x}_t} \approx \bar{x}(1 + \hat{x}_t)$$

Application of Loglinearization

Application: The equation

$$x_t + c_t = y_t$$

together with its steady state version

$$\bar{x} + \bar{c} = \bar{y}$$

deliver the dynamic relationship

$$\bar{x}\hat{x}_t + \bar{c}\hat{c}_t = \bar{y}\hat{y}_t$$

Example: RBC

Do it slowly for two equations:

- The resource constraint:

$$
\begin{aligned}
c_t + k_t &= y_t + (1-\delta)k_{t-1} \\
\bar{c}e^{\hat{c}_t} + \bar{k}e^{\hat{k}_t} &= \bar{y}e^{\hat{y}_t} + (1-\delta)\bar{k}e^{\hat{k}_{t-1}} \\
\bar{c}(1+\hat{c}_t) + \bar{k}(1+\hat{k}_t) &\approx \bar{y}(1+\hat{y}_t) + (1-\delta)\bar{k}(1+\hat{k}_{t-1}) \\
(\text{Note: } \bar{c} + \delta\bar{k} &= \bar{y}) \\
\bar{c}\hat{c}_t + \bar{k}\hat{k}_t &\approx \bar{y}\hat{y}_t + (1-\delta)\bar{k}\hat{k}_{t-1}
\end{aligned}
$$

Example: RBC

- The asset pricing equation:

$$
\begin{aligned}
\lambda_t &= \beta E_t\left[\lambda_{t+1} R_{t+1}\right] \\
\bar{\lambda} e^{\hat{\lambda}_t} &= \beta E_t\left[\bar{\lambda}\bar{R} e^{\hat{\lambda}_{t+1}+\hat{R}_{t+1}}\right] \\
1+\hat{\lambda}_t &\approx \beta \bar{R} E_t\left[1+\hat{\lambda}_{t+1}+\hat{R}_{t+1}\right] \\
(\text{ Note: } 1 &= \beta \bar{R}) \\
\hat{\lambda}_t &\approx E_t\left[\hat{\lambda}_{t+1}+\hat{R}_{t+1}\right]
\end{aligned}
$$

- On “ignored” Jensen terms: can also assume joint normality of logdeviations insteady. This changes the **steady state**, not the **dynamics**.

All loglinearized equations

#	Equation	Loglinearized
(i)	$\frac{1}{c_t} = \lambda_t$	$0 = \hat{c}_t + \hat{\lambda}_t$
(ii)	$A = \lambda_t (1-\theta) \frac{y_t}{n_t}$	$0 = \hat{\lambda}_t + \hat{y}_t - \hat{n}_t$
(iii)	$R_t = \theta \frac{y_t}{k_{t-1}} + 1 - \delta$	$0 = -\bar{R}\hat{R}_t + \theta \frac{\bar{y}}{\bar{k}} \left(\hat{y}_t - \hat{k}_{t-1} \right)$
(iv)	$y_t = \bar{\gamma} e^{z_t} k_{t-1}^{\theta} n_t^{1-\theta}$	$0 = -\hat{y}_t + z_t + \theta \hat{k}_{t-1} + (1-\theta)\hat{n}_t$
(v)	$c_t + k_t = y_t + (1-\delta) k_{t-1}$	$0 = -\bar{c}\hat{c}_t - \bar{k}\hat{k}_t + \bar{y}\hat{y}_t + (1-\delta)\bar{k}\hat{k}_{t-1}$
(vi)	$\lambda_t = \beta E_t[\lambda_{t+1} R_{t+1}]$	$0 = -\hat{\lambda}_t + E_t[\hat{\lambda}_{t+1} + \hat{R}_{t+1}]$
(vii)	$z_{t+1} = \rho z_t + \epsilon_{t+1}$	$z_{t+1} = \rho z_t + \epsilon_{t+1}$

Outline

Recursivity

- State variables are: k_{t-1}, z_t (or, alternatively, k_{t-1} and z_{t-1}).
- The dynamics of the model should be describable by a recursive law of motion (RLOM),

$$\begin{aligned} \lambda_t &= f_{(\lambda)}(k_{t-1}, z_t) \\ k_t &= f_{(k)}(k_{t-1}, z_t) \\ y_t &= f_{(y)}(k_{t-1}, z_t) \end{aligned}$$

etc.

Recursivity

- Assume that the RLOM is linear in the log-deviations,

$$\begin{aligned} \hat{\lambda}_t &= \eta_{\lambda k} \hat{k}_{t-1} + \eta_{\lambda z} z_t \\ \hat{k}_t &= \eta_{kk} \hat{k}_{t-1} + \eta_{kz} z_t \\ \hat{y}_t &= \eta_{yk} \hat{k}_{t-1} + \eta_{yz} z_t \end{aligned}$$

  etc. for coefficients $\eta_{\lambda k}$, $\eta_{\lambda z}$, etc.
- To make life simpler here, we shall try to reduce the system to only k and λ (one doesn't have to).

Simplify:

- Note: $\hat{y}_t = \frac{1}{\theta} z_t + \hat{k}_{t-1} + \frac{1-\theta}{\theta} \hat{\lambda}_t$
- Abbreviations:

$$\begin{aligned}
\alpha_1 &= \frac{\bar{y}}{\bar{k}} + (1-\delta) \\
\alpha_2 &= \frac{\bar{c}}{\bar{k}} + \frac{1-\theta}{\theta} \frac{\bar{y}}{\bar{k}} \\
\alpha_3 &= \frac{\bar{y}}{\theta \bar{k}} \\
\alpha_4 &= 0 \\
\alpha_5 &= 1 + (1-\theta) \frac{\bar{y}}{\bar{R}\bar{k}} \\
\alpha_6 &= \frac{\bar{y}}{\bar{R}\bar{k}}
\end{aligned}$$

Obtaining the solution

- We obtain the following **first-order two-dimensional stochastic difference equation**:

$$0 = -\hat{k}_t + \alpha_1 \hat{k}_{t-1} + \alpha_2 \hat{\lambda}_t + \alpha_3 z_t \quad (1)$$

$$0 = E_t[-\hat{\lambda}_t + \alpha_4 k_t + \alpha_5 \hat{\lambda}_{t+1} + \alpha_6 z_{t+1}] \quad (2)$$

$$z_t = \rho z_{t-1} + \epsilon_t \quad (3)$$

where z_t is an exogenous stochastic process.

Obtaining the solution

- Compare to the following first-order two-dimensional stochastic difference equation to be studied in the lecture on difference equations:

$$\begin{aligned} 0 &= -x_t + \alpha_1 x_{t-1} + \alpha_2 y_t + \alpha_3 z_t && (4) \\ 0 &= E_t[-y_t + \alpha_4 x_t + \alpha_5 y_{t+1} + \alpha_6 z_{t+1}] && (5) \\ z_t &= \rho z_{t-1} + \epsilon_t && (6) \end{aligned}$$

They are the same with $x_t = \hat{k}_t$, $y_t = \hat{\lambda}_t$.

The Method of Undetermined Coefficients

Postulate the **recursive law of motion**

$$\hat{\lambda}_t = \eta_{\lambda k}\hat{k}_{t-1} + \eta_{\lambda z} z_t \quad (7)$$

$$\hat{k}_t = \eta_{kk}\hat{k}_{t-1} + \eta_{kz} z_t \quad (8)$$

Plug this into equations (1) once and (2) "twice" and exploit $E_t[z_{t+1}] = \rho z_t$, so that **only the date-t-states $\hat{k}_{t-1}$ and z_t remain**,

$$\begin{aligned} 0 &= (-\eta_{kk} + \alpha_1 + \alpha_2\eta_{\lambda k})\hat{k}_{t-1} \\ &+ (-\eta_{kz} + \alpha_2\eta_{\lambda z} + \alpha_3) z_t \\ 0 &= (-\eta_{\lambda k} + \alpha_4\eta_{kk} + \alpha_5\eta_{\lambda k}\eta_{kk})\hat{k}_{t-1} \\ &+ (-\eta_{\lambda z} + \alpha_4\eta_{kz} + \alpha_5\eta_{\lambda k}\eta_{kz} + (\alpha_5\eta_{\lambda z} + \alpha_6)\rho) z_t \end{aligned}$$

Compare coefficients

On plugging in twice...

Plugging

$$\hat{\lambda}_t = \eta_{\lambda k}\hat{k}_{t-1} + \eta_{\lambda z} z_t, \quad \hat{k}_t = \eta_{kk}\hat{k}_{t-1} + \eta_{kz} z_t \quad \rho z_t = E_t[z_{t+1}]$$

twice into

$$\begin{aligned}
0 &= E_t[-\hat{\lambda}_t + \alpha_4 \hat{k}_t + \alpha_5 \hat{\lambda}_{t+1} + \alpha_6 z_{t+1}] \\
&= E_t[-(\eta_{\lambda k}\hat{k}_{t-1} + \eta_{\lambda z} z_t) + \alpha_4(\eta_{kk}\hat{k}_{t-1} + \eta_{kz} z_t) \\
&\quad +\alpha_5(\eta_{\lambda k}\hat{k}_t + \eta_{\lambda z} z_{t+1}) + \alpha_6 z_{t+1}] \\
&= E_t[-\eta_{\lambda k}\hat{k}_{t-1} - \eta_{\lambda z} z_t + \alpha_4 \eta_{kk}\hat{k}_{t-1} + \alpha_4 \eta_{kz} z_t) \\
&\quad +\alpha_5 \eta_{\lambda k}(\eta_{kk}\hat{k}_{t-1} + \eta_{kz} z_t) + (\alpha_5 \eta_{\lambda z} + \alpha_6) z_{t+1}] \\
&= (-\eta_{\lambda k} + \alpha_4 \eta_{kk} + \alpha_5 \eta_{\lambda k}\eta_{kk})\hat{k}_{t-1} \\
&\quad + (-\eta_{\lambda z} + \alpha_4 \eta_{kz} + \alpha_5 \eta_{\lambda k}\eta_{kz} + (\alpha_5 \eta_{\lambda z} + \alpha_6)\rho) z_t
\end{aligned}$$

Comparing coefficients

- On $\hat{k}_{t-1}$:

$$\begin{aligned} 0 &= -\eta_{kk} + \alpha_1 + \alpha_2 \eta_{\lambda k} \\ 0 &= -\eta_{\lambda k} + \alpha_4 \eta_{kk} + \alpha_5 \eta_{\lambda k} \eta_{kk} \end{aligned}$$

One gets the **characteristic quadratic equation**

$$0 = p(\eta_{kk}) = \eta_{kk}^2 - \left(\alpha_1 - \frac{\alpha_2}{\alpha_5} \alpha_4 + \frac{1}{\alpha_5} \right) \eta_{kk} + \frac{\alpha_1}{\alpha_5} \tag{9}$$

Solving the characteristic equation

Solutions:

$$\eta_{kk} = \frac{1}{2}\left(\left(\alpha_1 - \frac{\alpha_2}{\alpha_5}\alpha_4 + \frac{1}{\alpha_5}\right) \pm\sqrt{\left(\alpha_1 - \frac{\alpha_2}{\alpha_5}\alpha_4 + \frac{1}{\alpha_5}\right)^2 - 4\frac{\alpha_1}{\alpha_5}}\right) \tag{10}$$

Choose the stable root $\mid \eta_{kk} \mid < 1$. There is at most one stable root, if

$$\mid \eta_{kk,1}\eta_{kk,2} \mid = \mid \frac{\alpha_1}{\alpha_5} \mid > 1$$

With η_{kk}, calculate

$$\eta_{\lambda k} = \frac{\eta_{\lambda\lambda} - \alpha_1}{\alpha_2}$$

Comparing coefficients

- On z_t:

$$
\begin{aligned}
0 &= -\eta_{kz} + \alpha_2 \eta_{\lambda z} + \alpha_3 \\
0 &= -\eta_{\lambda z} + \alpha_4 \eta_{kz} + \alpha_5 \eta_{\lambda k} \eta_{kz} + (\alpha_5 \eta_{\lambda z} + \alpha_6)\, \rho
\end{aligned}
$$

Solution:

$$
\begin{aligned}
\eta_{\lambda_z} &= \frac{\alpha_4 \alpha_3 + \alpha_5 \eta_{\lambda k} \alpha_3 + \alpha_6 \rho}{1 - \alpha_4 \alpha_2 - \alpha_5 \eta_{\lambda k} \alpha_2 - \alpha_5 \rho} \\
\eta_{kz} &= \alpha_2 \eta_{\lambda z} + \alpha_3
\end{aligned}
$$

Outline

Step 5: Calculate impulse responses and (HP-filtered) moments

- Impulse responses: will be explained now.
- HP-filtered moments: will be discussed later.

Impulse Response Functions: response to a shock in z_t

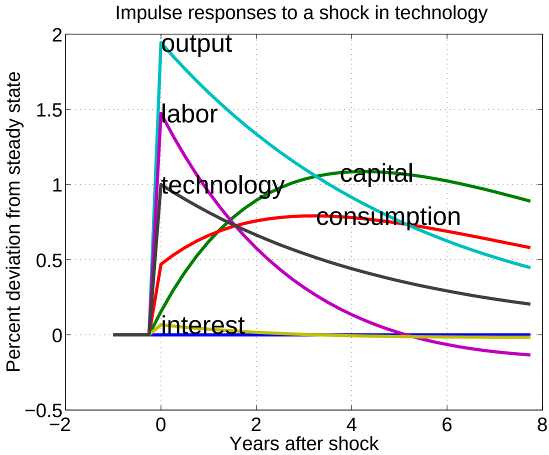
1. Set $z_0 = 0$, $\epsilon_1 = 1$, $\epsilon_t = 0, t > 1$
2. Calculate $z_t = \rho^t$
3. Set $\hat{k}_0 = 0$.
4. Calculate recursively

$$\hat{k}_t = \eta_{kk}\hat{k}_{t-1} + \eta_{kz}z_t$$

5. With that, calculate

$$\hat{\lambda}_t = \eta_{\lambda k}\hat{k}_{t-1} + \eta_{\lambda z}z_t$$

Results: Impulse Responses to shocks



Impulse Response Functions: response to an initial deviation of the state k_t from its steady state.

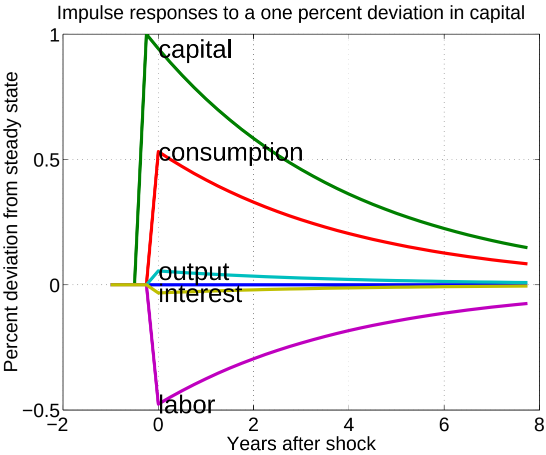
1. Set $z_t = 0, t \geq 1$
2. Set $\hat{k}_0 = 1$.
3. Calculate recursively

$$\hat{k}_t = \eta_{kk} \hat{k}_{t-1}$$

4. With that, calculate

$$\hat{\lambda}_t = \eta_{\lambda k} \hat{k}_{t-1}$$

Results: Impulse Responses to capital deviations



Outline

Recall: the loglinearized equations

#	Equation	Loglinearized
(i)	$\frac{1}{c_t} = \lambda_t$	$0 = \hat{c}_t + \hat{\lambda}_t$
(ii)	$A = \lambda_t (1-\theta) \frac{y_t}{n_t}$	$0 = \hat{\lambda}_t + \hat{y}_t - \hat{n}_t$
(iii)	$R_t = \theta \frac{y_t}{k_{t-1}} + 1 - \delta$	$0 = -\bar{R}\hat{R}_t + \theta \frac{\bar{y}}{\bar{k}} \left(\hat{y}_t - \hat{k}_{t-1} \right)$
(iv)	$y_t = \bar{\gamma} e^{z_t} k_{t-1}^{\theta} n_t^{1-\theta}$	$0 = -\hat{y}_t + z_t + \theta \hat{k}_{t-1} + (1-\theta)\hat{n}_t$
(v)	$c_t + k_t = y_t + (1-\delta) k_{t-1}$	$0 = -\bar{c}\hat{c}_t - \bar{k}\hat{k}_t + \bar{y}\hat{y}_t + (1-\delta)\bar{k}\hat{k}_{t-1}$
(vi)	$\lambda_t = \beta E_t[\lambda_{t+1} R_{t+1}]$	$0 = -\hat{\lambda}_t + E_t[\hat{\lambda}_{t+1} + \hat{R}_{t+1}]$
(vii)	$z_{t+1} = \rho z_t + \epsilon_{t+1}$	$z_{t+1} = \rho z_t + \epsilon_{t+1}$

A representation of the problem

There is an endogenous state vector x_t, size $m \times 1$, a list of other endogenous variables y_t, size $n \times 1$, and a list of exogenous stochastic processes z_t, size $k \times 1$. The equilibrium relationships between these variables are

$$\begin{aligned} 0 &= Ax_t + Bx_{t-1} + Cy_t + Dz_t \qquad (11) \\ 0 &= E_t[Fx_{t+1} + Gx_t + Hx_{t-1} + Jy_{t+1} + Ky_t + Lz_{t+1} + Mz_t] \\ z_{t+1} &= Nz_t + \epsilon_{t+1}; \quad E_t[\epsilon_{t+1}] = 0, \end{aligned}$$

where it is assumed that C is of size $l \times n$, $l \geq n$ and of rank n, that F is of size $(m + n - l) \times n$, and that N has only stable eigenvalues.

Example: RBC

Variables:

$$x_t = [\text{capital}] = [\hat{k}_t], \ y_t = \begin{bmatrix} \text{Lagrangian} \\ \text{consumption} \\ \text{output} \\ \text{labor} \\ \text{interest} \end{bmatrix} = \begin{bmatrix} \hat{\lambda}_t \\ \hat{c}_t \\ \hat{y}_t \\ \hat{n}_t \\ \hat{R}_t \end{bmatrix}$$

and

$$z_t = [\text{technology}] = [z_t]$$

Example: RBC

Matrices:

$$A = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\bar{k} \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ -\theta\frac{\bar{y}}{\bar{k}} \\ \theta \\ (1-\delta)\bar{k} \end{bmatrix}, C = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & \theta\frac{\bar{y}}{\bar{k}} & 0 & -\bar{R} \\ 0 & 0 & -1 & (1-\theta) & 0 \\ 0 & -\bar{c} & \bar{y} & 0 & 0 \end{bmatrix}$$

and

$$F = [0], G = [0], H = [0], J = [1, 0, 0, 0, 1],$$

$$K = [-1, 0, 0, 0, 0], L = [0], M = [0], N = [\rho]$$

Alternative representations 1

- Redefine the system as

$$\tilde{x}_t = \begin{bmatrix} x_t \\ y_t \end{bmatrix}, \tilde{F} = \begin{bmatrix} 0 & 0 \\ F & J \end{bmatrix}, \tilde{G} = \begin{bmatrix} A & C \\ G & K \end{bmatrix},$$
$$\tilde{H} = \begin{bmatrix} B & 0 \\ H & 0 \end{bmatrix}, \tilde{L} = \begin{bmatrix} 0 \\ L \end{bmatrix}, \tilde{M} = \begin{bmatrix} D \\ M \end{bmatrix},$$

- The system can then be rewritten as a second-order stochastic matrix difference equation,

$$0 = E_t \left[F\tilde{x}_{t+1} + \tilde{G}\tilde{x}_t + \tilde{H}\tilde{x}_{t-1} + \tilde{L}z_{t+1} + \tilde{M}z_t \right]$$
$$z_t = Nz_{t-1} + \epsilon_t;\ E_{t-1}[\epsilon_t] = 0 + \tilde{\epsilon}_t$$

Alternative representations 2

- Redefine the system as

$$\tilde{x}_t = \begin{bmatrix} x_t \\ y_t \\ z_t \end{bmatrix}, \tilde{\epsilon}_t = \begin{bmatrix} 0 \\ 0 \\ \epsilon_t \end{bmatrix},$$

$$\tilde{F} = \begin{bmatrix} 0 & 0 & 0 \\ F & J & L \\ 0 & 0 & 0 \end{bmatrix}, \tilde{G} = \begin{bmatrix} A & C & D \\ G & K & M \\ 0 & 0 & -I_k \end{bmatrix}, \tilde{H} = \begin{bmatrix} B & 0 & 0 \\ H & 0 & 0 \\ 0 & 0 & N \end{bmatrix}$$

- The system can then be rewritten as a second-order stochastic matrix difference equation,

$$0 = E_t \left[F\tilde{x}_{t+1} + \tilde{G}\tilde{x}_t + \tilde{H}\tilde{x}_{t-1} \right] + \tilde{\epsilon}_t$$

Alternative Representations 3

- E.g. per stacking,

$$\check{x}_t = \begin{bmatrix} \tilde{x}_t \\ \tilde{x}_{t-1} \end{bmatrix}$$

etc., one can even rewrite the system as a first-order stochastic matrix difference equation,

$$0 = E_t \left[F\check{x}_{t+1} + G\check{x}_t \right] + \check{\epsilon}_t$$

- Here, one needs to keep in mind, that some entries in x_t are predetermined, i.e. already fixed as of $t-1$.
- This representation is often used, e.g. in Blanchard-Kahn, Farmer, many others.

Various Representations 4

- Various representations appear in the literature.
- Which representation is most convenient? That depends on the solution approach.
- The “complicated” first representation has the advantage of focussing on a small numer of state variables.